Maths HUBS Yorkshire Ridings

Strengthening Reasoning

A Handbook for Teachers

Contents

What do the children think?

"Reasoning is fun because it makes your brain really work! The challenge is so exciting!"
Y6 pupil

"I love reasoning because it is challenging but, it's easy to find an approach and get there in the end if you really put your mind to it!"
Y6 pupil

"This is the best type of maths because it makes you think about what you are doing in different ways'
Nathan Year 6

"Maths is more interesting when you have to find it out for yourself '
Jack Year 6

"I now know that the answer is not really the important thing in maths"
Charlie Year 5

"I like being able to solve maths in my own way ….I'm getting better at explaining what I mean'
Ruby Year 6

What do teachers think?

"The children are engaged, challenged and begin to think deeply about every problem they face."

"Giving children time to explore and explain in their own way gives you an insight into their thinking"
Year 4 Teacher

"Setting an expectation that maths should be justified has led to much deeper thinking and mathematical discussion……. the children learn so much from each other"
Year 6 teacher

"When the children are reasoning, it gives me chance to talk to them much more, finding misconceptions and challenging them further."

"Being more rigorous with questioning in my maths sessions has given me a much better idea about who really understands the maths behind the solution and why it works"

"I understand my children's thought processes a lot more when I provide them one or two reasoning problems than if I had asked them to complete an entire page of basic calculations."
Year 6 teacher.

National Curriculum requirements

KS1, 2 and 3 National Curriculum Aims:

The national curriculum for mathematics aims to ensure that all pupils:

- **reason mathematically** by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language

KS3 National Curriculum: Working mathematically

Through the mathematics content, pupils should be taught to:

Reason mathematically

- extend their understanding of the number system; make connections between number relationships, and their algebraic and graphical representations
- extend and formalise their knowledge of ratio and proportion in working with measures and geometry, and in formulating proportional relations algebraically
- identify variables and express relations between variables algebraically and graphically
- make and test conjectures about patterns and relationships; look for proofs or counter-examples
- begin to reason deductively in geometry, number and algebra, including using geometrical constructions
- interpret when the structure of a numerical problem requires additive, multiplicative or proportional reasoning
- explore what can and cannot be inferred in statistical and probabilistic settings, and begin to express their arguments formally.

GCSE Assessment Objective AO2

Reason, interpret and communicate mathematically

- make deductions, inferences and draw conclusions from mathematical information
- construct chains of reasoning to achieve a given result
- interpret and communicate information accurately
- present arguments and proofs
- assess the validity of an argument and critically evaluate a given way of presenting information

Reasoning: the Journey from Novice to Expert

Developing reasoning skills with young learners is a complex business. They need to learn to become systematic thinkers and also acquire the ability to articulate such thinking in a clear, succinct and logical manner. In many classrooms more progress is being made with developing the systematic thinking than with the elegant communication. There needs to be equal emphasis on both these aspects of reasoning and in both we need to consider progression. What would we expect from a novice reasoner as opposed to an expert reasoner? How can we help young learners to progress to expert level?

Progression in reasoning

At NRICH we see a five-step progression in reasoning: a spectrum that shows us whether children are moving on in their reasoning from novice to expert. Children are unlikely to move fluidly from one step to the other, rather flow up and down the spectrum settling on a particular step that best describes their reasoning skills at any one time.

<u>Step one</u>: Describing: simply tells what they did.

<u>Step two</u>: Explaining: offers some reasons for what they did. These may or may not be correct. The argument may yet not hang together coherently. This is the beginning of inductive reasoning.

<u>Step three</u>: Convincing: confident that their chain of reasoning is right and may use words such as, 'I reckon' or 'without doubt'. The underlying mathematical argument may or may not be accurate yet is likely to have more coherence and completeness than the explaining stage. This is called inductive reasoning.

<u>Step four</u>: Justifying: a correct logical argument that has a complete chain of reasoning to it and uses words such as 'because', 'therefore', 'and so', 'that leads to' ...

<u>Step five</u>: Proving: a watertight argument that is mathematically sound, often based on generalisations and underlying structure. This is also called deductive reasoning.

[Extract from nrich article: https://nrich.maths.org/11336]

- Is a planned, coherent programme for developing systematic thinking embedded in your scheme of learning?
- Is this matched by an equal focus on developing elegant communication?
- Are your learners' reasoning skills moving progressively through the five steps above?
- How do teachers encourage learners to move more fluidly up the spectrum if they have settled on a particular step?

We need to value and promote reasoning explicitly, persistently, consistently and frequently and, in particular, help children to develop complete chains of reasoning. This aspect of mathematics will help us to deepen and extend our higher attainers as we take them onto generalisations and proof, whilst focusing on the same mathematical content.

[Extract from nrich article: https://nrich.maths.org/11336]

The importance of discussion in strengthening reasoning

"Talking is central to our view of teaching mathematics formatively . . . Providing opportunities for students to express, discuss and argue about ideas is particularly important in mathematics. Through exploring and unpacking mathematics, students can begin to see for themselves what they know and how well they know it."

Hodgen & William (2006): Mathematics inside the black box

"The national curriculum for mathematics reflects the importance of spoken language in pupils' development across the whole curriculum – cognitively, socially and linguistically. The quality and variety of language that pupils hear and speak are key factors in developing their mathematical vocabulary and presenting a mathematical justification, argument or proof. They must be assisted in making their thinking clear to themselves as well as others and teachers should ensure that pupils build secure foundations by using discussion to probe and remedy their misconceptions."

Mathematics Programmes of Study: Key stage 3, DfE

"… schools whose mathematics work was outstanding had a consistently higher standard of teaching…. The schools focused on building pupils' fluency with, and understanding of, mathematics. Pupils of all ages and abilities tackled varied questions and problems, showing a preparedness to grapple with challenges, and explaining their reasoning with confidence."

Mathematics: Made to Measure, p7, Ofsted, 2012

Some strategies for generating productive mathematical discussion

- *Plan questions that provoke thinking and that are worthy of exploration*

- *Allow adequate thinking time before inviting responses*

- *Think - pair - share*
  After personal thinking time, learners discuss ideas with a partner. This gives them opportunity to expand and rehearse their explanations. Teacher then asks one pair for their ideas which are then built on by subsequent pairs

- *"Ask them, don't tell them"*
  Draw all of the mathematics from the learners and co-construct all mathematical ideas. Follow through all incorrect suggestions to identify why they are wrong, and ask learners to comment on and critique others' suggestions. Model mathematical thinking as you work through a problem. Learners need to see the thinking as an organic process that requires constant re-evaluation whilst moving towards the solution to a problem.

- *Develop questioning styles through 'Ping-pong' and 'French cricket' to 'Volleyball'*
  Questioning sequences often involve only the teacher and a single learner (analogous to a game of ping pong). These have their value when the teacher is pushing for amplification or depth, but other learners may not gain anything from this unless they are actively listening or thinking. In the French cricket analogy the ball is returned to a different bowler, corresponding to the teacher asking another learner to comment on or add to what they have just heard. By inviting further comment, more students become involved and the mathematical argument is extended. In a game of volleyball the ball is kept in the air as it moves between different players. This corresponds to learners commenting on others' responses without intervention or moderation by the teacher.

- *Focus on the process, not the 'answer'*
  Mathematical thinking is about the journey towards a solution rather than the solution itself. Avoid presenting standard methods as *faits accomplis*, but instead guide learners towards formulating them through questions such as "What could we do next?", "What might work here?", "What will happen if we follow this suggestion?", "Has this brought us closer to the solution?", etc.

- *Compare and contrast different approaches*
  When inviting suggestions from learners it is important that these are all valued and explored, and that they are used to build the mathematical dialogue. Evaluate the suggested approaches with questions such as "Will this method always work?", "Who's methods do we prefer - and why?", "Can you think of an occasion when A's Method will be more efficient than B's Method?", etc.

Teaching for relational understanding

"… there are in current use two meanings of [the word understanding]. These [are distinguished] by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding'."

Skemp, R. R., 1976: 'Relational understanding and instrumental understanding', Mathematics Teaching, 77, 20–26

"Mastery is not just being able to memorise key facts and procedures and answer test questions accurately and quickly. It involves knowing 'why' as well as knowing 'that' and knowing 'how'. It means being able to use one's knowledge appropriately, flexibly and creatively and to apply it in new and unfamiliar situations"

Askew, Bishop, Christie, Eaton, Griffin and Morgan, 2015: 'Teaching for Mastery', OUP / NCETM, p6

- *Use diagrams to clarify structure*
  Use carefully chosen diagrams to build conceptual understanding before moving to more formal methods. Eg for dividing a quantity in a given ratio. A good visual representation may also inform the solution process such as when solving linear equations.

- *Move freely between different representations*
  Ensure learners become familiar with the same ideas presented in different forms (eg fraction and percentage or pie chart, bar chart and frequency diagram). Discuss which representation is best suited to a particular problem, and why.

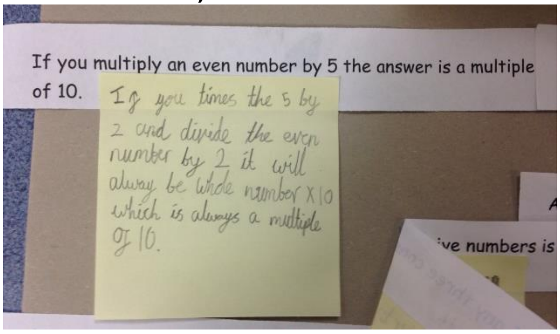
- *Devise and use questions that combine different areas of maths*
  Learners need to understand their mathematics as a complex web of connected ideas. Teaching topics in isolation prevents this. In particular, there is a natural place for the use of variables and algebraic expression in most topics as generalisations are disovered and expressed.

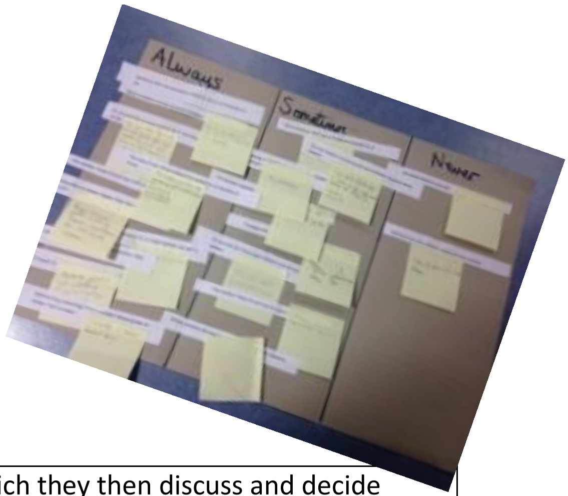
Generic Activities that support some principles of reasoning

Most of these activities are listed alphabetically and described in detail in the next pages

Revealing and exploring underlying structure	Fill the Venn Change one aspect Diving for depth Give me an example … and another… Make up a question (and mark scheme) Odd one out Reverse the question Reversed two-way tables
Surfacing and resolving misconceptions	Always - sometimes - never Concept Cartoons Dear Doctor Diagnostic questions Diving for depth Spot the mistake
Building relational understanding and making connections	Change one aspect Diving for depth Easy and hard Topic mats
Moving to generality and introducing variables	Same and different Give me an example... and another… I like - I don't like On the spot generalisation
Constructing chains of reasoning and reasoning deductively	Muddled proofs Spot the mistake Two column proofs
Making and testing conjectures and reasoning inductively	I like - I don't like On the spot generalisation Reversed two-way tables

Always, sometimes, never





Description:	Learners are given a statement which they then discuss and decide whether it is sometimes, always or never true. If ‘sometimes’ then they determine the conditions for which it is or isn’t true and give some examples. If they think the statement is always true or never true then learners are asked to justify or prove it.
Why use this activity?	Supports construction of mathematical arguments and can be used to introduce formal proofs. Draws attention to boundary conditions and focuses on underlying properties of structures.
More examples:	A straight line with a positive gradient passes through the first (positive) quadrant. A quadrilateral has two obtuse internal angles. A shape with three lines of symmetry also has rotational symmetry. A multiple of 12 is also a multiple of 8. $2x + 3 = 3x + 3$ $2x + 6 = 2(x + 3)$ $2x + 3 = 2w + 3$
Where could it be used?	All topics, all years. Useful at the start of a topic as a hook to promote discussion. Also works well whenever a new object is introduced and its properties are to be explored.
What do I need to be mindful of?	Don’t use the words ‘always’ or ‘never’ in the statements that are to be discussed. A single counterexample shows that a statement cannot always be true, but further exploration is needed to test whether it may sometimes be true.
Notes:	

“At first I thought I knew the answer was ‘never’ but when I looked at it properly I realised it could work”
Calum Year 6.

“The best part about it is when someone else has a different idea and you have to try and prove them wrong”
Joel Year 6.

Always, sometimes, never pupils' work

All prime numbers are odd.

- always
- sometimes
- never ✓

Explain your choice.

Not all prime numbers are odd because 2 is a prime number.
2 and 1 are the only factors in 2.

A prime number is a number that has factors of 1 and the prime number itself.

Square numbers cannot be prime numbers.

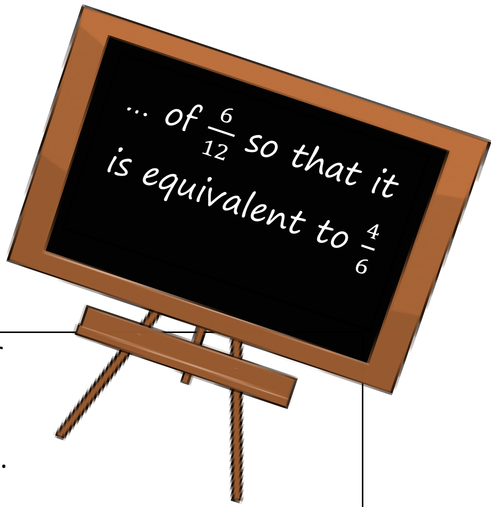
- always ✓
- sometimes
- never

Explain your choice.

1 X 1 = 1 (1 factor)
2 X 2 = 4 (3 factors)
3 X 3 = 9 (3 factors)
4 X 4 = 16 (5 factors)
5 X 5 = 25 (3 factors)
6 X 6 = 36 (5 factors)

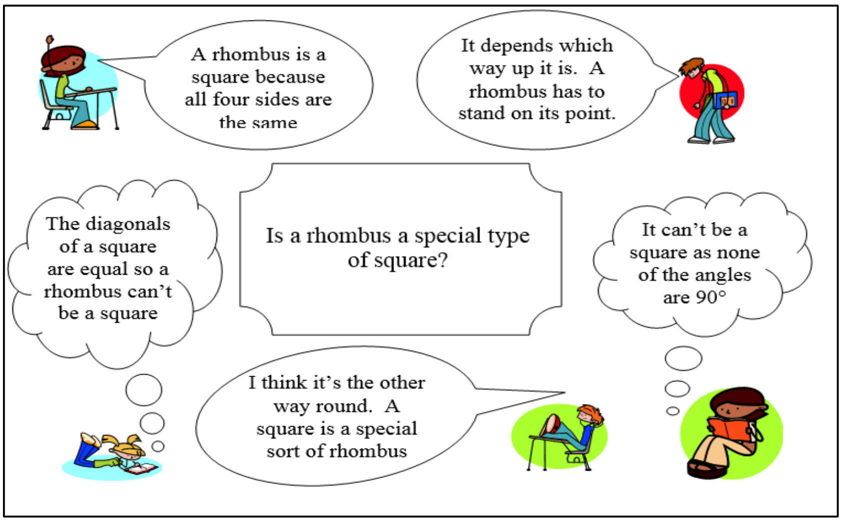
NO square numbers can be a prime number because they always have themself, 1 and the number to create the prime number (square root).

Change one aspect



Description:	Learners alter a given statement, object or mathematical item in some way so that it matches a given condition. Any aspect of the original may be modified.
Why use this activity?	Draws attention to underlying structures, and the effect of variation, giving deeper understanding of the role of variables. Can lead to generalisation, eg the statement will be true for all numbers greater than … Altering different aspects of the original may also lead to exploration of inverses and reciprocals.
Examples:	Change one aspect of … … y=2x+3 so that the line passes (4, 10) … $^{2}/_{3}$ x $^{3}/_{4}$ so that the answer is greater than 1 … $2x^2 + 3x + 4 = 0$ so that x=2 is a solution
Where could it be used?	In any topic where the component parts of a formula or object are to be explored and understood.
What do I need to be mindful of?	Always ask whether any other aspects can be altered so that learners become aware that more than one dimension of variation is possible.
Notes:	

Concept Cartoon



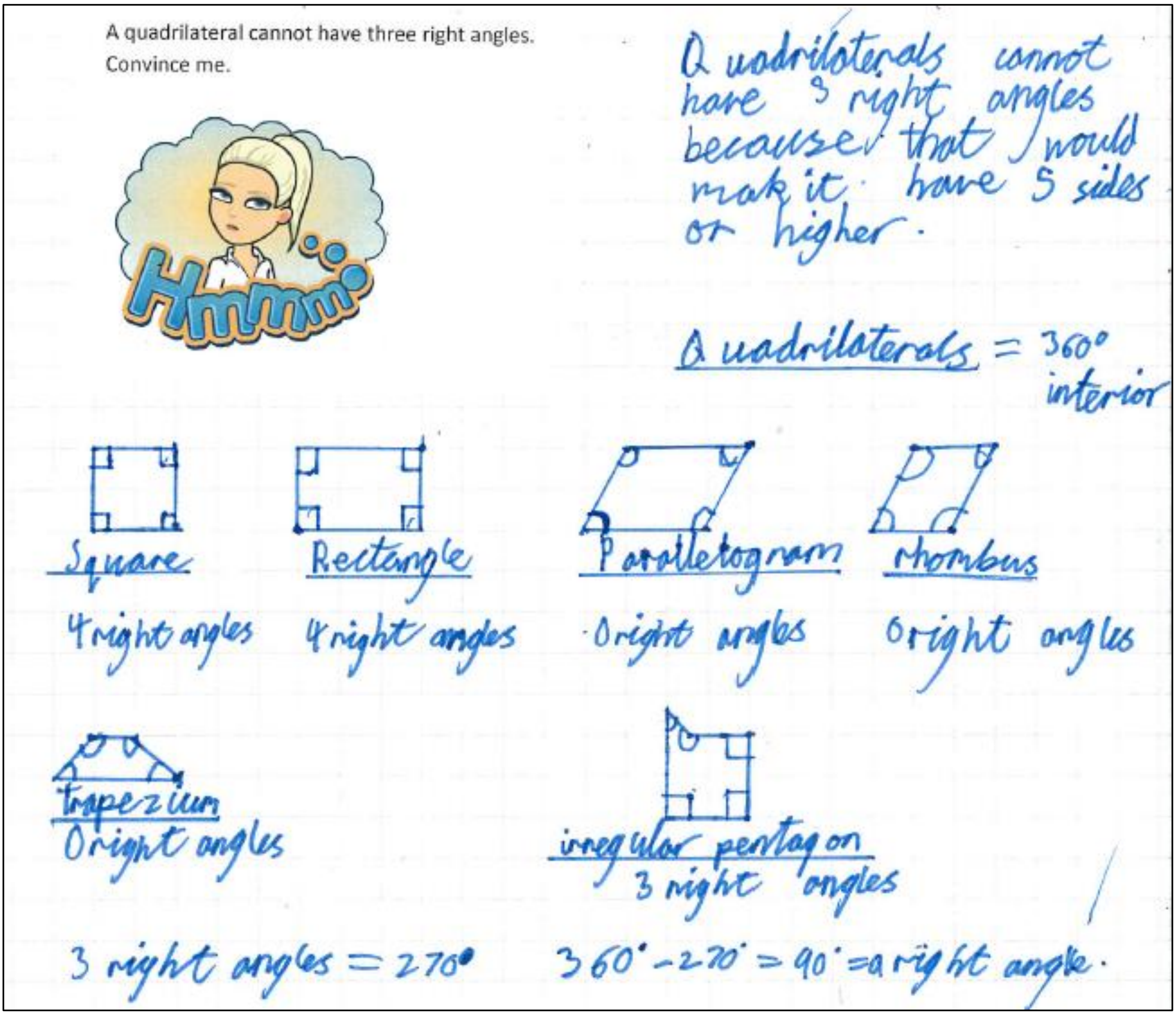
Description:	A question is presented in the centre of a page. Around this are a series of thought bubbles showing some possible learner responses. These will include some correct answers, some misconceptions, and ideally some that reflect the thinking process. Learners consider each response and explain why the person may have thought this, what misconception is shown, and what advice they would give to a person who thinks this.
Why use this activity?	Raises potential misconceptions in a depersonalised, non-threatening way. Encourages learners to engage with others' thinking.
Where could it be used?	Works with all topics, all age groups, wherever misconceptions occur. Can be used at the start of a topic to assess prior understanding, or at the end of a topic to draw the learning together
What do I need to be mindful of?	All misconceptions shown on the Concept Cartoon must be fully discussed and resolved otherwise learners may leave with their existing wrong idea reinforced.
Notes:	

Convince a cynic

Description:	After working on a problem learners are asked to convince a partner that their solution is correct. The partner plays the role of 'Devil's Advocate', challenging every statement and demanding justification. No assumptions are allowed to remain unchallenged.
Why use this activity?	Helps learners to frame mathematical arguments with increasing rigour. Supports extended reasoning sequences.
Examples:	A quadrilateral cannot have three right angles. A multiple of nine is divisible by three. $\sin\theta° = \cos(90-\theta°)$
Where could it be used?	Use in any situation where learners are required to justify mathematical ideas. A useful strategy for supporting their move through the steps of reasoning (see nrich article, p4), particulalrly if learners are initially asked to convince themselves, then to convince a friend, then to convince an expert. This should raise the level of rigour and quality of argument at each stage.
What do I need to be mindful of?	Learners must provide mathematical reasons for their observations and not simply rely on hunches. Encourage use of appropriate connectors to sequence the argument such as 'therefore', 'because', 'I know that … and so ….', etc
Notes:	

Convince a cynic - Pupil’s work

> *“I love proving my teacher, or my friends wrong (or sometimes right!), and debating throughout the problem!”*



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Dear Doctor

Dear Doctor

When I have to solve an equation like 2(3x + 5) = 12 I always multiply out the bracket first.

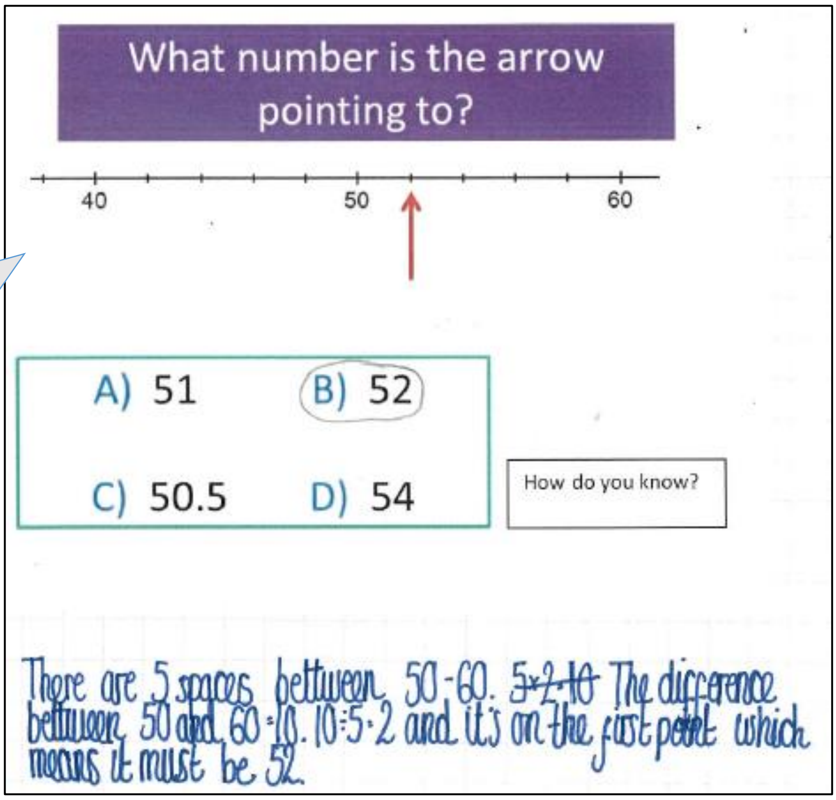
My friend says you should divide by the number outside the bracket instead. What do you think is the best way?

Algie Burr

Description:	Learners take on the role of Agony Aunt or Agony Uncle for the problems page of a maths magazine. They are given a problem that has been submitted by a reader, and in pairs they then write a reply
Why use this activity?	Encourages learners to explore potential misconceptions and engage with others' thinking. Can be used to assess depth of learners' understanding. Strengthens literacy in mathematics and gives practice at presenting arguments clearly.
Example:	Dear confused child, You wouldn't need to divide by 20 because it would get you 5% of the number. This is because if you divide the number by 20 you are halving 10% of it which would result in you getting a smaller fraction. Instead you would have to find 10% of the number first, which is 1/10 of the number. You find this be dividing by the denominator and multiplying by the numerator. In this case, you would multiply by 2 so that you would get 2/10 (⅕). Yours Sincerely, Doctor Bradley Vickers 
Where could it be used?	Any topic where misconceptions are likely. Use with groups who have a focus on strengthening literacy skills. Learners' replies can make a good poster display and also provide a useful opportunity for peer-assessment of other's suggestions. Is the argument clear? Does it go deep enough? Does it provide a full enough explanation? Should any other points be included?
What do I need to be mindful of?	Don't allow the activity of writing to preclude mathematical thinking. Perhaps discuss with learners what their solutions will be before committing to paper.
Notes:	

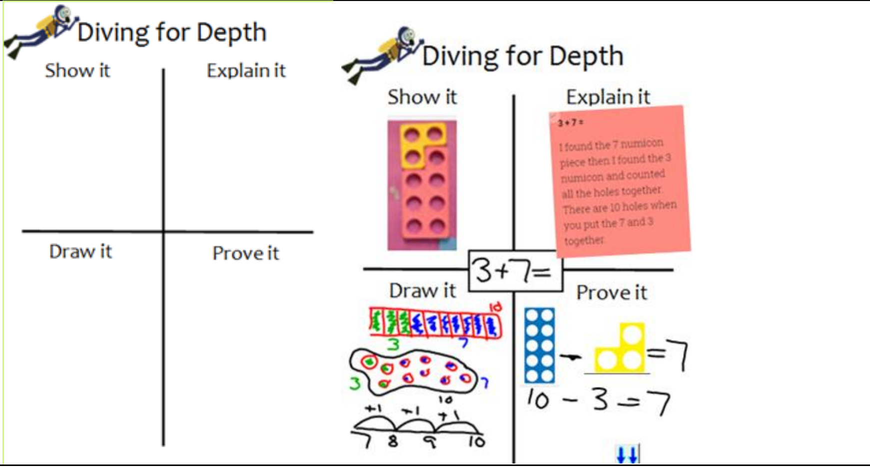
Diagnostic questions

"I like having multiple choices as it made me test the others to see if they work. It's like all the mistakes we could make!"
Melissa

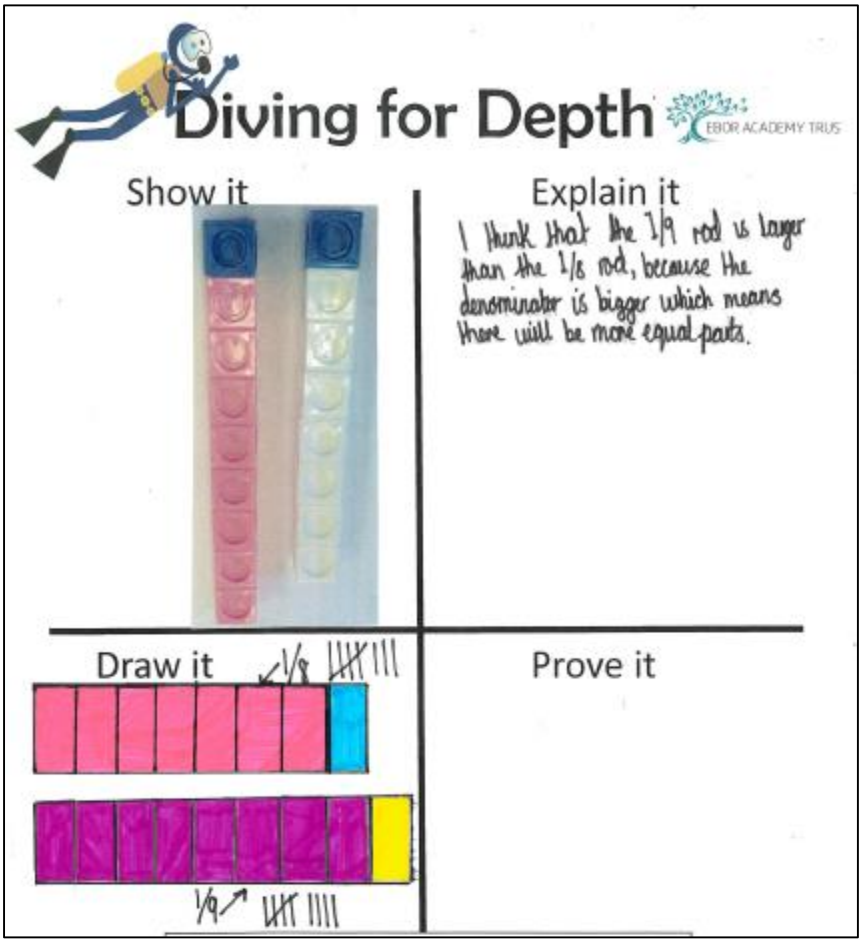


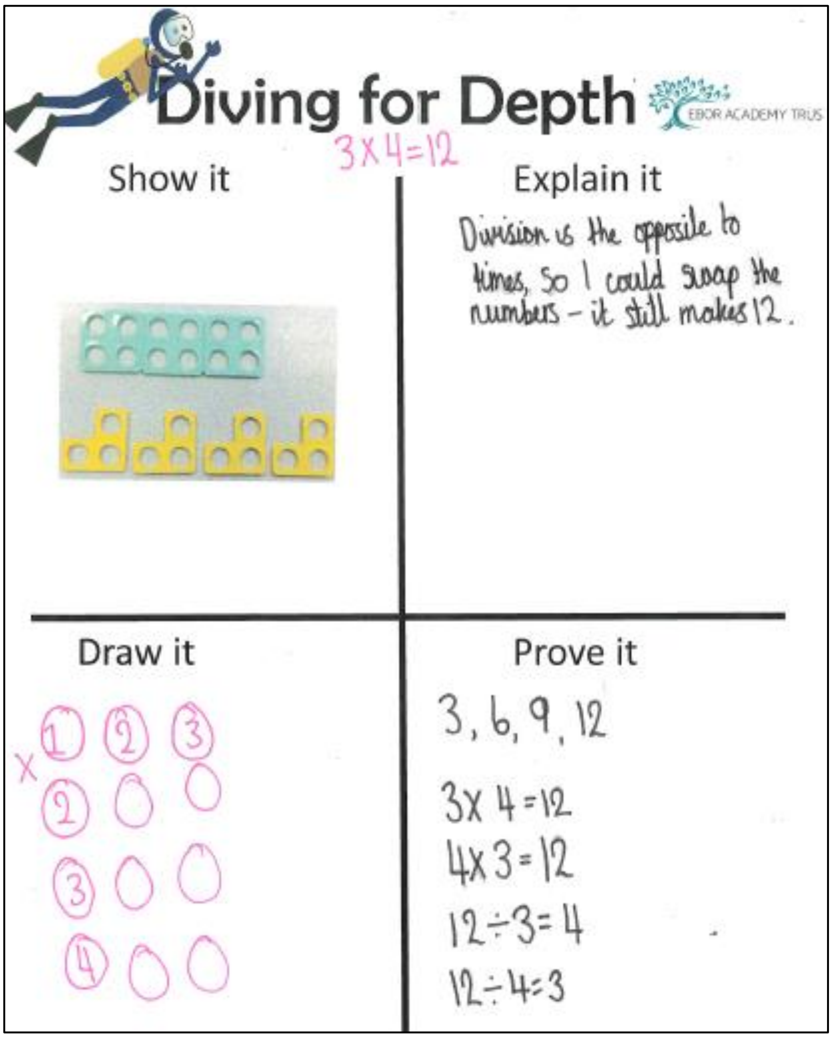
Description:	A question is given, along with four possible answers. Learners decide which answer is correct, but also discuss how the incorrect answers might have been obtained.
Why use this activity?	Useful formative assessment for identifying (diagnosing) misconceptions. Builds metacognition as learners explain the wrong thinking in the incorrect solutions. Learners can also be asked to create their own diagnostic question and potential answers. This will draw their attention to likely errors and enhance understanding of the *process* required to solve similar problems.
Example:	Example above from https://diagnosticquestions.com/
Where could it be used?	Every topic, every year group
What do I need to be mindful of?	All incorrect solutions must be explained and corrected. Reasons why the right answer is correct need to be fully discussed.
Notes:	

Diving for depth

Description:	This resource can be used in many ways. It is a tool to aid the Concrete – Pictorial - Abstract approach to learning. A question is presented to the children, which requires a response to each of the titles. Show it = Concrete representation: ie: numicon, unifix, multilink etc. Draw it = Representational/pictorial: ie: bar method, tally marks, groupings etc. Explain it = reasoning using maths vocabulary. Prove it/Apply it/Inverse it = To encourage the child to un-pick the question structure.
Why use this activity?	The sheet gives learners chance to show, draw, explain and prove their working out. It helps teachers to thoroughly understand the thinking behind the child's answer, alongside highlighting any misconceptions.
Example:	
Where could it be used?	This activity works well in wave one teaching either as the main body of a lesson, as a quick starter/plenary activity or independent task to assess the children's' full understanding.
What do I need to be mindful of?	The sheet must be modelled well by the teacher. A variety of different ways to 'show' or 'draw' a calculation must be explored by the class for this activity to have full effect. Teachers must be mindful that there can be many ways to represent each problem/calculation and difference is encouraged within the classroom; this gives the class chance to discuss the differences and the effectiveness of each. All misconceptions must be fully addressed when they occur. It is in no way essential that the sheet is followed through in the given order, or that each box is always used.
Notes:	

Diving for depth - Pupils' work





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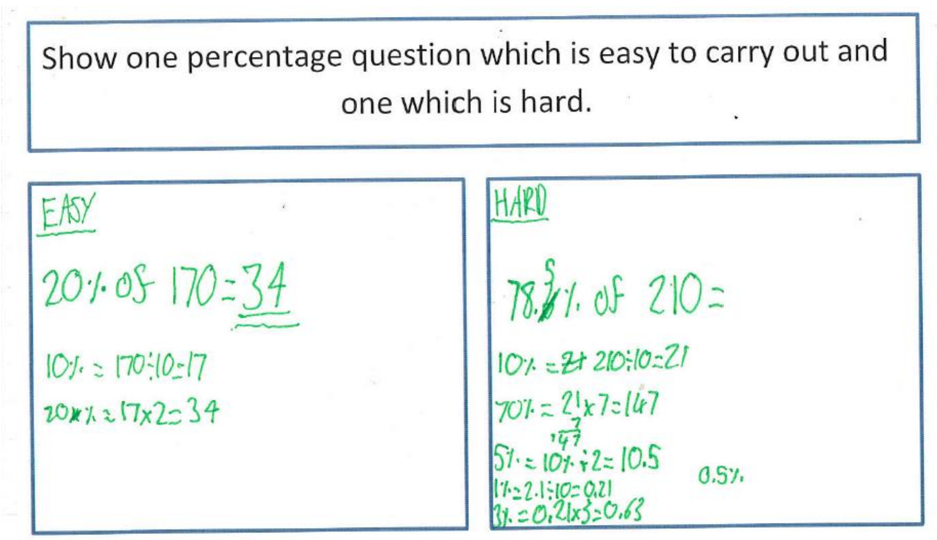
Easy and hard

Description:	Ask learners to create an easy, a medium and a hard example of a given mathematical object / problem. They may then solve their own examples or pass to other pairs to solve and review.
Why use this activity?	By thinking about what makes a problem hard, learners focus on the structure of the mathematical object that they are dealing with and the process of solution. A useful assessment activity to gauge the depth of learners' understanding. Different pairs of learners may classify the same problem differently, which provokes discussion and subsequent peer-tutoring.
Examples:	A fraction that is easy (medium, hard) to simplify A quadratic equation that is easy (medium, hard) to solve A percentage calculation that is easy (medium, hard) to carry out
Where could it be used?	Towards the end of a topic when reviewing different question types and solution methods. Also very useful as a revision strategy, particularly if learners then write a set of instructions on how to solve each type. This metacognitive approach placed greater emphasis on the structural differences.
What do I need to be mindful of?	Perceived difficulties may be to do with the nature of the numbers involved rather than with a more complex underlying mathematical structure. Eg: a learner may consider x + 2 = 7 an easier equation to solve than x + 3.28 = 5.91 (which may well be true!) although these have the same structure. Ensure that discussion of relative difficulty is based on the structural differences between examples.
Notes:	



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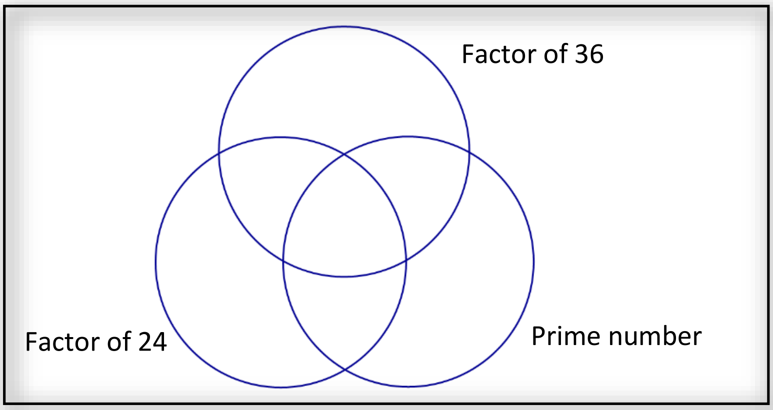
Fill the Venn

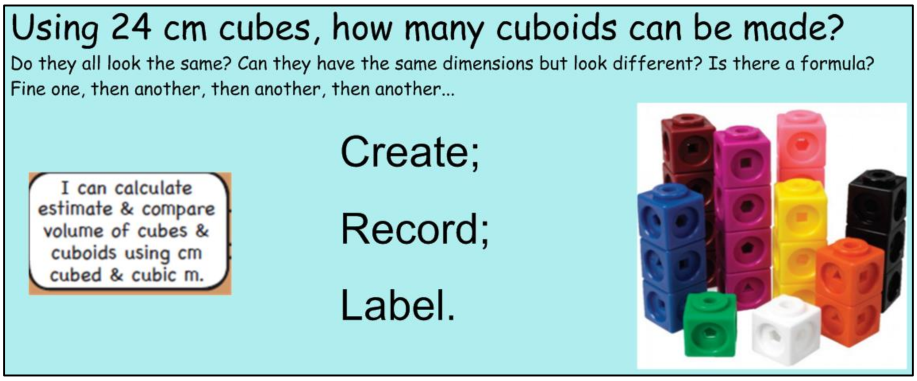


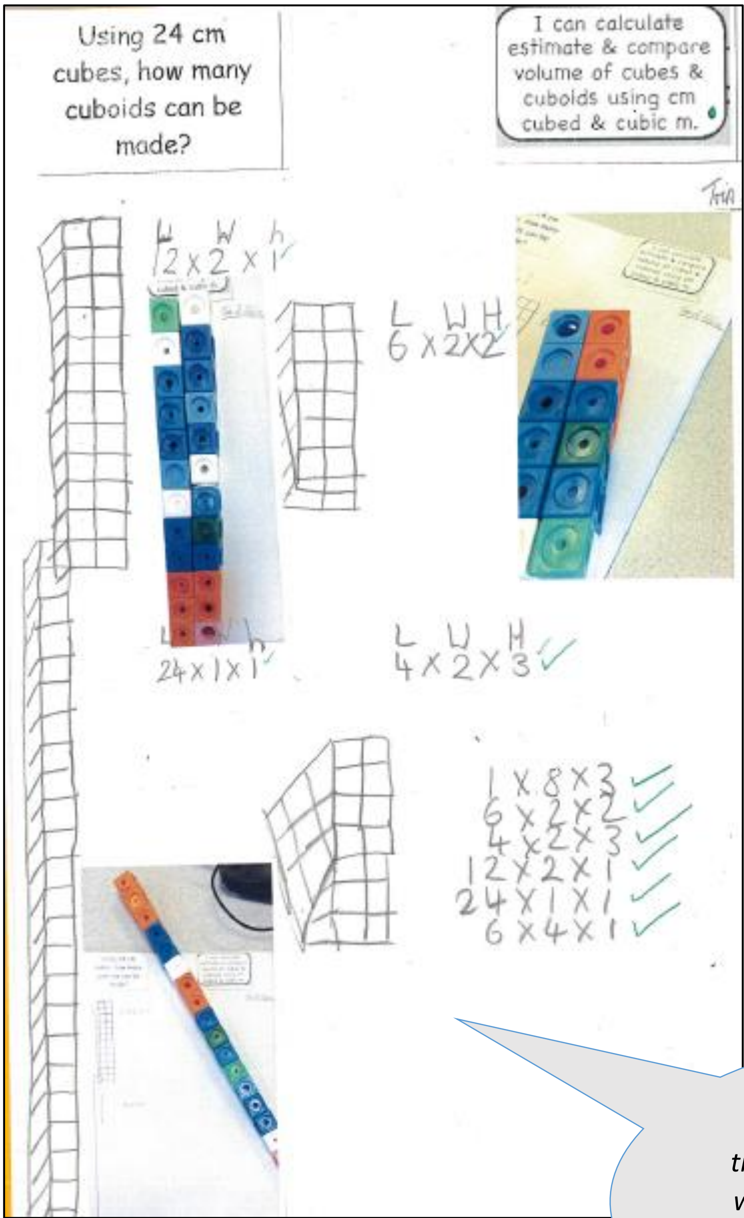
Description:	Learners are given an empty Venn diagram showing two or three intersecting sets. Each of the sets is labelled with a property. They then fill in examples for each part of the diagram that match the appropriate combination of properties. If it is not possible to find an example for a particular subset then learners must explain why. Can they generalise if there are several possible examples for one of the subsets?
Why use this activity?	Provides deeper exploration of different properties of objects, particularly those which can be shared or which are mutually exclusive. Encourages definition of objects in terms of their properties and allows consideration of the question 'Why is this not a ……' as well as the more usual 'What is a ……'
Example:	
Where could it be used?	Any topic where clarification of properties is required. Eg different shapes, number types, linear graphs, trig functions, integrals Use to deepen understanding after objects have been defined by their properties. A useful formative review activity at the end of a lesson or topic.
What do I need to be mindful of?	Check, through discussion, that the subsets are filled in accurately. Ensure learners explain *why* some regions may be empty - and that they aren't left blank just because learners can't find an object for the region.
Notes:	See also 'Same and Different'

Give me an example, …. and another…

Description:	Ask learners to write down an example of something, then another, then another. Then ask for an interesting example, and then one that they think nobody else will have written. Now discuss how they created their unusual examples, and what all of the examples have in common. This will then lead to generalisation of the structure and opens the way to introducing variables to express the general case.
Why use this activity?	To reveal underlying mathematical structures. To introduce variables and algebraic representation.
Examples:	A rectangle with perimeter 20cm [which will lead to the generalisation of edges with length w and $10 - w$]. A fraction that simplifies to ½. This will lead to the generalisation $^{n}/_{2n}$
Where could it be used?	All topics, all years
What do I need to be mindful of?	That the generalisation is correct, for instance that addition is not used when generalising a proportional change (eg there may be a temptation to generalise $^{3}/_{6}$ to $^{n}/_{n+3}$ as a fraction equivalent to ½. Note that this could then provoke discussion about why $^{n}/_{(n+n)}$ is equivalent to $^{n}/_{2n}$ and to ½)
Notes:	

Give me an example, …. and another…





"I like this method because it's like trying out different predictions, then when you have found one, you have to keep thinking which gets more and more difficult!"

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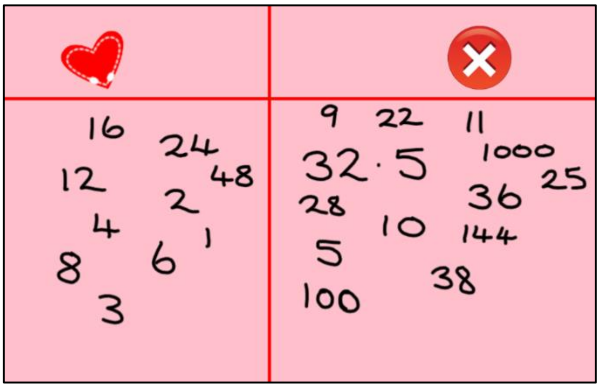
I like - I don't like



I Like	I don't like
square	sphere
parallelogram	cone
cube	circle
triangle	ellipse
	cylinder

Description:	The teacher (or a learner) thinks of a rule / attribute / property that an object may possess. Learners then deduce the rule by testing particular examples. These are recorded on a chart / whiteboard: 'I Like' if the object has the property, otherwise 'I don't like'
Why use this activity?	Helps learners recognise and distinguish between properties of objects. Supports development of inductive thinking. Learners make and test conjectures, and are likely to be exposed to cognitive conflict as their initial ideas are modified.
Examples:	Defining property is "shapes that have only straight edges" Defining property is "numbers that are factors of 36" Defining property is "fractions that are equivalent to ½" Defining property is "trig functions which have a value of $\sqrt{3}/2$" Defining property is "Lines that pass through the point (2, 4)"
Where could it be used?	All years, all topics. Particularly effective when learners make up their own rules and challenge each other.
What do I need to be mindful of?	Don't let learners call out the rule when they think they know it. Instead ask for an example that they know you will like, and one that they know you will definitely not like.
Notes:	See also 'What's the same, what's different' and 'On The Spot Generalisation' from Thinkers

"It makes you think more because you have to try different theories to figure out the rule"

Justification Template

STEPS TO SUCCESS

Look for angles between the planks in the gate.

Decide if the angles are more or less than a right angle.

Highlight different angles using a key.

To find reflex angles , you can just use the exterior angle of an acute angle.

Angles:

How many different types of angle can you see?

FACT BOX

An acute angle = 0-89 degrees.

R/A = 90 degrees

An obtuse angle = 90 - 180 degrees.

A reflex angle = 181 - 360 degrees.

Circle = 360 degrees

Description:	This is an adaptable template which is used to develop pupils' reasoning and explanation skills. A question or challenge is added to the centre of the sheet and the pupils have to break down and identify all of the associated mathematical skills and knowledge. Pupils are required to think more about the problem and the process of arriving at the solution, than the actual answer itself.
Why use this activity?	This activity will develop pupils' understanding of their own mathematical processes and will highlight links between different areas of maths. When used effectively learners will be encouraged to identify all relevant mathematical skills and knowledge that can be used to find a solution. They will create a list of 'Facts' and, a set of 'Steps to Success'. These two aspects will help the pupil to demonstrate their mathematical processes and will act as a guide to the solution. This template can also be used an opportunity for pupils to explore a range of solutions to the same problem.
Examples:	Example challenge questions : 0.6 is the answer, what is the question …. 18 x 5 – how many ways can this be solved? Calculate the shaded area of this shape…. (Almost any mathematical problem can be used, providing it gives learners an opportunity to draw together a range of mathematical facts and explain how they help)
Where could it be used?	This can be used across almost all areas of maths, from general calculation methods and algorithms to geometry and shape. It is particularly useful for those pupils who struggle to explain their own mathematical process or those who have a limited range of strategies.
What do I need to be mindful of?	Pupils will need to be supported through the process. The purpose and the importance of generating the fact box information may need to be explored and practised prior to working independently.
Notes:	

"It helps because you start thinking about all the different pieces of maths that can help you …… the problem starts getting easier'
Jacob Year 5.

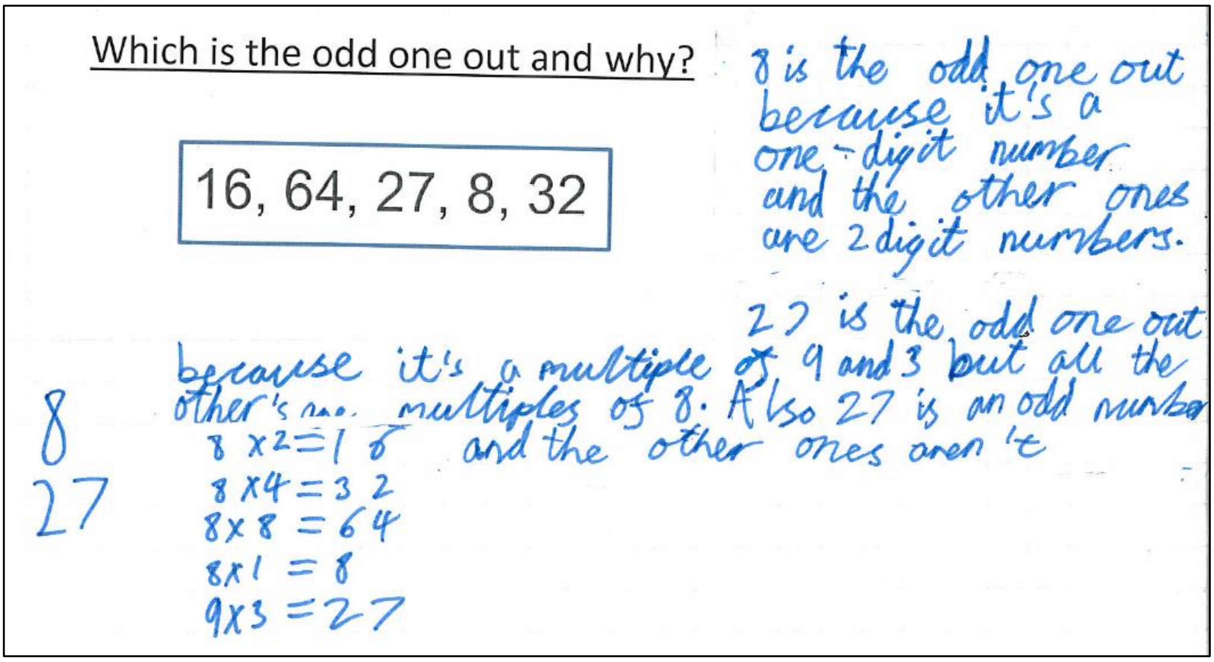
"The one we did showed us that there are lots of different ways to solve a calculation, not just the usual way……. our group came up with eight different strategies!" Jessica

Muddled proofs

Description:	A formal proof or a chain of reasoning is written out, one statement per line, each on separate cards. The cards are then sorted into the correct order by learners who then compare their order with others'. When agreement has been reached the cards are then stuck into books or onto a poster, with an explanation of what mathematical operation has been used to move from card to card.
Why use this activity?	When building chains of reasoning. When learners need support in sequencing parts of an argument. When moving to formality and written proof.
Example:	$2(m + n + 1)$ is even $2m + 2n + 2 = 2(m + n + 1)$ $(2m + 1) + (2n + 1) = 2m + 2n + 2$ Every odd number can be written as $2m + 1$ The sum of two odd numbers is an even number Every even number is twice another number $2(m + 1)$ is an even number
Where could it be used?	When learners are secure in explaining, convincing and justifying, and need support in moving towards more formal proof.
What do I need to be mindful of?	Ensure that learners understand how each line of proof is derived from the previous one.
Notes:	

Odd one out

Description:	Three items are given and learners are asked to select an odd one out and give reasons for their choice. Different selections can then be evaluated and justified. Challenge learners to find a reason for *each* of the objects to be the odd one out. When they have made their choice, ask learners to make up a fourth object that will pair with their odd one out to give two sets of two objects - two that possess the attribute and two that don't.
Why use this activity?	Supports classification of objects and leads to deeper understanding of their properties. The creation a fourth object draws attention to defining properties and underlying structure (as in Reverse the Question)
Examples:	Three shapes: Square, Rhombus, Rectangle Three whole numbers: 15, 25, 36 Three straight lines: y=x + 3, y=x + 2, y=3-x
Where could it be used?	Any topic where objects share some properties but not others
What do I need to be mindful of?	The reason for the odd one out must be a mathematical attribute that the selected object possesses, and not a negative such as 'It's the only one that hasn't already been the odd one out'
Notes:	



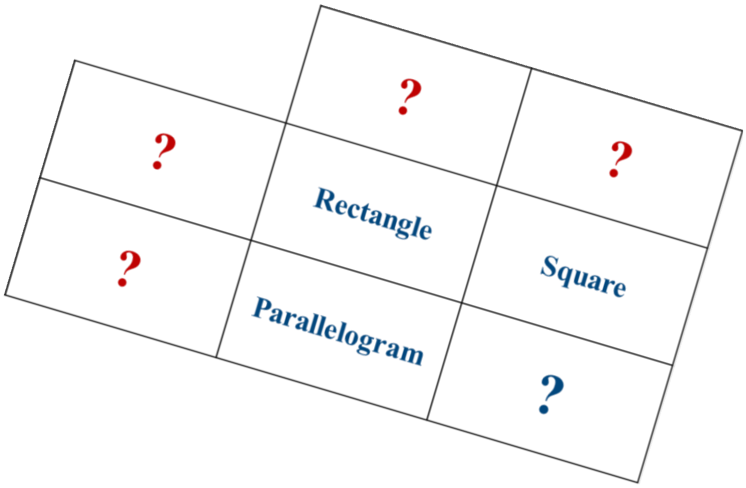
On the spot generalisation

Description:	Learners are given a single mathematical example or statement, and asked to generalise from this. They then test their conjecture by providing more examples and checking that they are valid.
Why use this activity?	Encourages learners to make and test conjectures. They look at the statement and form a hunch as to what might be happening. This hunch is then tested and formalised into a generalisation if it is correct.
Examples:	6 x 7 is an even number 2^5 - 1 is a prime number 1/9 = 0.1111111…
Where could it be used?	In any situation where we are wanting to explore the question 'Why does this happen?'
What do I need to be mindful of?	Generalisations must be fully tested and then explained, justified or proved. It is unusual (and dangerous) to generalise from a single example, but this exploits the most basic mathematical question 'What might be going on here?' The single statement provokes curiosity to explore further.
Notes:	

Reverse the question

Description:	Within a given topic, learners are provided with an answer and then asked to construct a question that leads to the given solution. Many different 'questions' are likely to be produced. These can then be compared and contrasted (What's the same, what's different about them) to reveal underlying structures and to clarify the different solution processes.
Why use this activity?	'Reverse the question' prevents instrumental 'jump through the hoop' learning and reproduction of standard methods. It instead helps learners to focus on the solution process in order to work backwards, drawing on aspects of inverse operations. It also develops an appreciation of variation and the effect of changing the value of one aspect of a question.
Examples:	A linear equation with solution $x = 2$ A fraction that simplifies to ¾ Two numbers that add to 27
Where could it be used?	Every topic, every year group
What do I need to be mindful of?	Check that learners remain focused on the topic in hand and don't invent questions from other topics that yield the same solution! Also prompt learners to create a range of questions of increasing difficulty so that they move beyond single-step solutions.
Notes:	

Reversed two-way table



Description:	A 2-way table is presented with some cells filled in, but without row or column labels. Learners decide what the missing properties might be, and find possible items to fill the empty cells.
Why use this activity?	Strengthens inductive thinking. Encourages deeper understanding of mathematical objects through consideration and comparison of their defining properties. Provokes discussion of what an object is not, as well as what it is.
Example:	
Where could it be used?	All age groups and any topics where items can be defined by (more than one) characteristic.
What do I need to be mindful of?	There is often more than one combination of properties that will give a valid solution. Consider these and ask whether there are any others. How do we know? Are there any combinations of properties that would make it impossible to fill one of the cells?
Notes:	See also 'Same and Different' and 'I like - I don't like'

Spot the mistake

9 The number of goals scored in 15 hockey matches is shown in the table.

Number of goals	Number of matches
1	2
3	1
5	5
6	3
9	4

Calculate the mean number of goals scored.

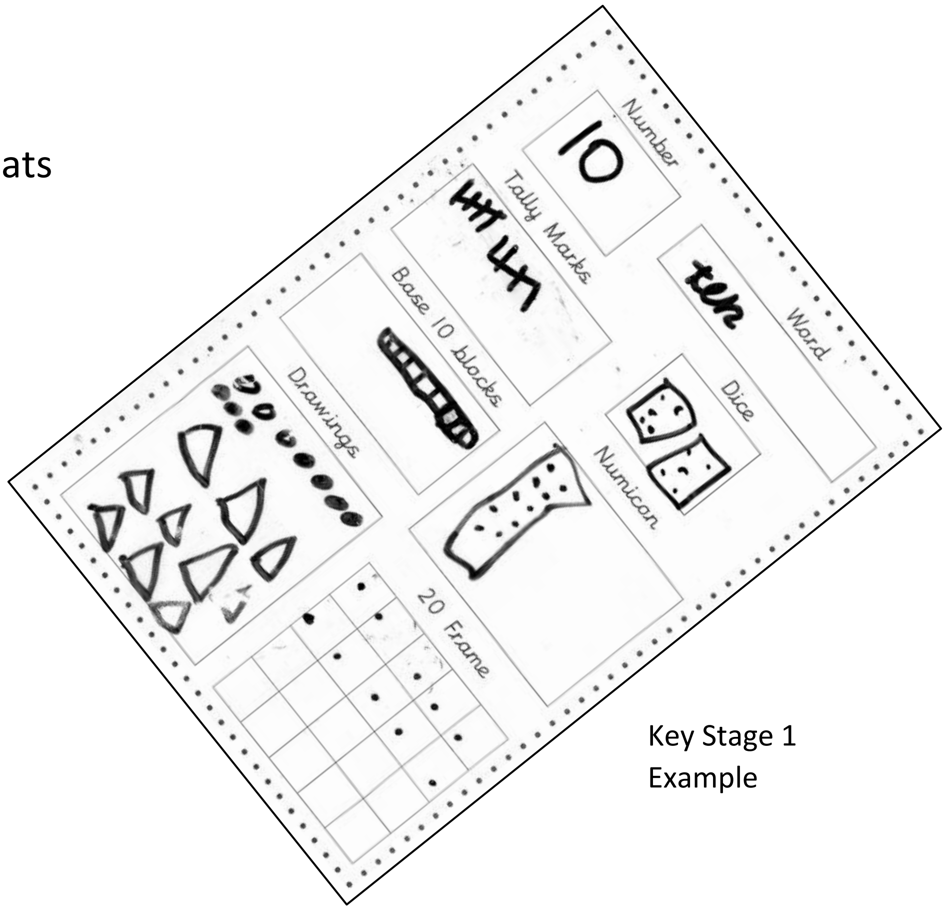
$1 + 3 + 5 + 6 + 9 = 24$

$24 \div 5 = 4.8$

Answer 4.8 goals *(3 marks)*

Description:	An incorrect worked example is given to learners. They identify and correct the mistake, then discuss why the mistake may have been made.
Why use this activity?	If the incorrect examples are constructed carefully they will draw attention to potential misconceptions. The activity encourages engagement and criticism of others' thinking processes and raises learners' awareness of where they might go wrong in tackling similar problems.
Example:	
Where could it be used?	All topics, but particularly useful for revision before exams
What do I need to be mindful of?	All errors must be identified and explained and learners must recognide that the example given is incorrect
Notes:	

Topic Mats



Key Stage 1
Example

Description:	A laminated A3 template made up from a set of blank tables / charts / diagrams is provided. When filled in, these will all show the same information but in different forms. Groups of learners are then given the information for one diagram, and they are then required to reproduce this in the equivalent forms. If different groups are given different starting points then the finished mats can be compared - what's the same, what's different?
Why use this activity?	Encourages fluency in switching between multiple representations. Helps to build relational understanding. Draws attention to potential connections between seemingly unrelated topics.
Examples:	Different forms of presenting data Different algebraic forms of equation of a line or curve Sequences - graphs - functions Fractions - decimals - percentages
Where could it be used?	Any topic that offers multiple representations of the same information.
What do I need to be mindful of?	
Notes:	

Some questioning stems to support reasoning activities

Before the task begins……

What do you already know?

Which facts do you think you will need?

Which pieces of the problem make sense to you and which parts are confusing?

Do you think the answer is obvious? Why?

Which words and information are important?

Where do you think we should start?

During the task…….

If you broke your method into steps, what would they be?

Why did you choose this method? Is it working in the way you expected?

Has anything surprised you?

How do you know that this is correct?

Are there any other methods you could use?

What are other members of your group doing – why have they chosen to do it that way?

Can you explain the challenge in your own words?

Can you convince me that you are correct?

Following / Towards the end of the task…….

Can you create and solve a problem similar to this one?

Do you have enough evidence to prove that this is correct?

Does this always work?

What knowledge have you used to help you with this?

Can you explain why your first answer was incorrect?

What would happen if …..?

Could you explain this to somebody who doesn't understand the problem?

Questions for getting started

What exactly am I trying to find or do?

What clues have I got?

What *do* I know?

Is there another way to tackle the problem?

Have I seen anything like this before?

Can I represent the information differently?

Can I work out *anything* that might help get to the solution?

What topic is this question to do with?

What can I do with the information I have?

Is there something about a similar problem that could help me?

Can I see any patterns?
Can I work at this systematically?

Can I split the problem into stages?

What facts do I know about this topic?

Will a diagram help?
Can I add anything to the diagram that gives me more information?

MathsHUBS Yorkshire Ridings

References and further reading

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Anne Watson, Jeff Jones, Dave Pratt	Key Ideas in Teaching Mathematics	OUP / Nuffield Foundation	0199665518	http://www.nuffieldfoundation.org/key-ideas-teaching-mathematics
Jeremy Hodgen, Dylan Wiliam	Mathematics Inside the Black Box	Kings College / GL Assessment	0708716873	http://www.gl-assessment.co.uk/products/mathematics-inside-black-box
Stephanie Prestage, Pat Perks	Adapting and Extending Secondary Mathematics Activities	David Fulton	185346712X	https://www.routledge.com/products/9781853467127
John Mason, Sue Johnstone-Wilder	Designing and Using Mathematical tasks	Tarquin / Open University	1899618651	http://www.tarquingroup.com/product.php?SKU_Code=1259
John Mason, Leone Burton, Kaye Stacey	Thinking Mathematically	Prentice Hall / Pearson	0273728917	
Malcolm Swan	Improving Learning in Mathematics: strategies and challenges	DfES Standards Unit	184478537X	Downloadable from STEM library: https://www.stem.org.uk/elibrary/collection/2933/improving-learning-in-mathematics
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Student Achievement Division, Ontario Schools	Asking effective questions in math	Ontario Ministry of Education		http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf

Members of the Yorkshire Ridings Maths Hub Reasoning Work Group

This handbook has been developed by a group of primary and secondary teachers who have worked together at the Yorkshire Ridings Maths Hub. The booklet forms part of a wider professional development package available to teachers across the YR Hub region.

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